The University of Northern Colorado

Mathematics Contest

Learning – Growing – Bonding

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Introduction

For the past 15 years the Department of Mathematical Sciences at the University of Northern Colorado has sponsored a mathematics competition open to all students in grades 7-12 throughout the state of Colorado. This competition is designed to offer a unique opportunity to mathematically talented students by developing, deepening and extending their experiences in mathematics. Through similar competitions at Stanford University in the 1950's, Santa Clara University (1960 –) and the University of New Mexico (1970 – 1990) this UNC competition can trace its philosophical roots back to the Hungarian Eötvös competitions during the time period 1894-1905. Nationally, there are many varied and successful competitions that address the needs of a wide variety of talented students offering vehicles through which these students can continue to develop problem solving skills and to explore new and challenging regions of mathematics. The AHSME, the Michigan Mathematics Prize Competition and the U.S. Mathematical Olympiad are among the largest and most prestigious. Equally important, at such competitions mathematically gifted students can find a friendly playing field on which to engage socially as well as competitively and where their special abilities can shine through these new bonded friendships. The UNC competition is dedicated to serving the needs of these students.

The goals of the contest

Consistent with its predecessor competitions, the UNC contest has the following six goals:

1. Offer a unique educational challenge to all interested secondary students.
2. Recognize and reward talented students for their extraordinary achievements.

3. Provide an opportunity for UNC faculty to cooperatively engage in an educational endeavor involving secondary school teachers, parents, and students.
4. Recruit and advise.
5. Draw attention to fundamental mathematical themes in the secondary curriculum.
6. Provide a meaningful forum for discussion. A large number of talented mathematics students can gather and share their ideas about mathematics.

What makes this contest different?

- All students in grades 7–12 in the state of Colorado are eligible. A student need not be selected or prescreened. Often an academically unrecognized student will perform well on the contest.
- All students take the same exam.
- The contest consists of two rounds, separated by three months.
- Each 3 hour round consists of 10 or 11 essay type questions. There are no multiple choice questions.
- The first round is graded jointly by secondary teachers and UNC faculty.
- The problems are paired – a theme is introduced in the first round and is then further developed in the final round.
- A solutions seminar for teachers and parents is conducted while the contest is underway.

The nature of the competition

This contest is designed as an “extended” year round learning experience. The November First Round consists of 10–11 open-ended questions designed to mildly challenge and to create intrigue. This round is mailed out to all secondary schools in the state; our long term goal is to nurture a collaborative attitude with the secondary mathematics teachers. Scheduling flexibility has always been of utmost concern; our partial solution to this problem has been to allow the 3 hour exam to be taken during any available time slot over a three day period which includes a weekend. All schools return solution sheets to UNC, where a team of UNC faculty, secondary teachers, and some graduate students conduct a grading session which typically lasts about half a day. One of the most valuable aspects of the contest occurs during this session. Discussions about various answers, solutions, and the questions themselves often lead to deeper understanding of the secondary curriculum, expectations, goals of certain high school programs, how to deal with the gifted, etc. Graded reports are returned to participating schools along with an announcement of those invited back for the February Final Round. The returnees always include a representative number of students from each of the grades 7 through 12.

The problems on the First Round vary in difficulty. The main intent is to encourage a very broad range of students to persist with a feeling that they can be successful in their attempts at solving nonroutine problems. We deliberately avoid problems that involve trigonometry or calculus so that a talented 7^{th} or 8^{th} grade participant can more readily compete on an equal basis with an 11^{th} or 12^{th} grader. Typically four or five of the First Round problems foreshadow a tougher, more challenging version on the Final Round. It was at this stage that we observed students seriously becoming involved in learning mathematics. Between the First and Final Rounds students collaborate with one another and with their teachers, often sharing their thoughts on websites, in an effort to deepen their understanding of underlying topics that could be generalized or built upon later. This pairedness aspect rewards planning and aggressiveness. Topics on both rounds include arithmetic and geometric sequences, the Pythagorean theorem, Pascal like triangles, divisibility properties, triangular numbers, recurrence relations, geometry, probability, factoring, basic counting procedures and pattern recognition.

While the Final Round is quite challenging, it is inevitable that some student will achieve nearly a maximum score. From the top 10% participating in the Final Round we identify the top 25 students and also approximately 20 Honorable Mentions. The top 25 are honored along with parents and teachers at an April Awards Banquet. Prizes include graphing calculators, plaques and several mathematics texts for each student.

The Final Round is conducted during a Saturday session on campus; a solutions seminar is conducted simultaneously at a nearby site. We observed many years ago that parents and teachers were often reluctant to discuss the problems with their child or student, fearing that they would expose their own inadequacies or inexperience with mathematics. The solutions seminar partially solved this confidence issue. Together with the moderator, parents and teachers worked on the exam questions and derived (sometimes jointly) solutions that were presented to all. At the conclusion of this session all were empowered (and actually were eagerly excited) to engage with the students as the competition concluded. In recent years there have been between 60–70 in attendance at this seminar.

The paired problems

Several of the paired problems from the past decade where the easier version foreshadows a more difficult extension are presented next.

FIRST ROUND: What is the smallest positive multiple of 75 that consists of just 0's and 1's?

FINAL ROUND: What is the smallest positive multiple of 225 that consists of just 0's and 1's?

FIRST ROUND: Determine the number of positive integral solutions to

$$\frac{1}{m}+\frac{1}{n}=\frac{1}{2^3\cdot 3^2}$$

FINAL ROUND: Determine the number of positive integral solutions to

$$\frac{1}{m}+\frac{1}{n}=\frac{1}{2^{50}\cdot 3^{40}}$$

FIRST ROUND: In the number spiral, give the next three numbers to the right of 11, 2, 1, 6, 19.

13	14	15	16	17			
12	3	4	5	18			
11	2	1	6	19			
10	9	8	7	20			
		…	22	21			

FINAL ROUND: In the number spiral, give with proof a general formula for the numbers in the sequence, 1, 6, 19, … that appear as part of row three as shown.

FIRST ROUND: The odd number 7 can be expressed as $16 - 9 = 4^2 - 3^2$, a difference of two squares. Express each of 17 and 83 as the difference of two squares.

FINAL ROUND: (a) Demonstrate that every odd number $2n + 1$ can be expressed as a difference of two squares.

(b) Demonstrate which <u>even</u> numbers can be expressed as a difference of two squares.

FIRST ROUND: The three sides of a triangle have integer lengths *a*, *b*, and *c* satisfying the inequality $a \le b \le c$. How many noncongruent triangles are there for $c = 10$?

FINAL ROUND: Determine a formula for the number of noncongruent triangles for general *c*, and explain your reasoning.

<u>Be tenacious and creative</u>

Many problems in mathematics submit themselves to a variety of solutions. Through our solutions seminar and correspondence with teachers and students we encourage the search for "nice" solutions, creative solutions. Consider the following question: How many positive integers have their digits in strictly increasing order? Of the many possible solutions, one of the winners zeroed in on the following very succinct and beautiful solution: Since any subset of $\{1,2,3,4,5,6,7,8,9\}$ except the empty set will correspond to an "increasing" integer, the answer is $2^9 - 1$. Attacking problems like this with a calculator-like approach simply ruins the day. Our admonition to be creative echoes what Albert Einstein once implied. *We all have a brain. It's what we do with it that matters*. We must train our brain to be creative. For example, there are only three pure colors, but look at what Michelangelo did with them. There are but seven musical notes, but listen to what Chopin, Beethoven and Mozart did with these seven notes.

Some data

Participation in the contest has grown from 140 students in 1992 to over 2600 in 2004; the 2006–2007 contest attracted about 1800. The number of schools supporting participation has increased from the initial six to over 75 recently. Data collected over the past twelve years revealed that of the top 25 winners each year 18% were women. In 2007, for the first time in the 15 year history, a female participant achieved first place. Also, during this time frame, 40% of those who placed in the top 10 were students from grades 8, 9, and 10. And, a somewhat surprising result, 38% of the time first place was achieved by a student in either the 8^{th}, 9^{th}, or 10^{th} grade. These underclassmen are quite talented.

Where are they now?

Winners have, with great consistency and few exceptions, gone on to distinguish themselves academically at the very highest levels. A recent query to parents, teachers and the winners themselves produced the following information. Mark graduated from Rice with degrees in Mathematics and Chemical Engineering and is starting a PhD at UT Austin in CE. Brian is working on a PhD in Mechanical Engineering at Stanford. Anne is in medical school in N.Y. Brenton has a BS in M.E. from MIT and is starting graduate work. T.J. earned an MS in Electrical Engineering at Stanford, now starting his PhD program. Ben graduated with a degree in E.E. from Cornell, completed medical school at U. of Michigan. Tom completed an MS at U. of Michigan and is now in their aerospace engineering PhD program. Both Steven and Richard, brothers, graduated from Harvard, and then on to Columbia. One completed law school, the other medical school. Kevin is at Stanford, Andy at CU Boulder, math majors both. Karl is completing his PhD in mathematics at U. of Wisconsin. His research has been characterized as "*beautiful and totally unexpected*", tracing back to Ramanujan's work.

Tom reports: "*The math and critical thinking skills developed in participating in math competitions such as the UNC competition have certainly helped me in my career*." Mark adds: "*The contest at UNC… had a profound affect on my academic career. Good contest problems involve looking beyond the surface appearance of patterns in order to find a deeper understanding. This outlook has helped me approach many problems in a variety of fields from a unique perspective*." Seth – "*I enjoyed the test immensely*."